Chapter 9

Statics and Torque

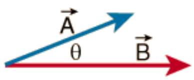
The Dot Product (Scalar Product)

- The dot product of two vectors can be constructed by taking the component of one vector in the direction of the other and multiplying it times the magnitude of the other vector. This can be expressed in the form:

The Dot Product (Scalar Product)

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta$$

Calculation



$\vec{A}$ denotes vector

A denotes the magnitude of the vector.

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

where

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$$

Applications

The Dot Product use (Scalar Product)

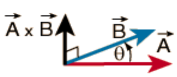
Geometrically, the scalar product is useful for finding the direction between arbitrary vectors in space. Since the two expressions for the product:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z = A\,B\cos\theta$$

The Cross Product

The magnitude of the vector product of two vectors can be constructed by taking the product of the magnitudes of the vectors times the sine of the angle (<180 degrees) between them. The magnitude of the vector product can be expressed in the form:

$$|\vec{A}\times\vec{B}|_{magnitude} = A\,B\sin\theta$$



$\vec{A}\times\vec{B}$ is perpendicular to both A and B

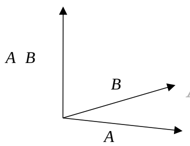
Calculation

The Cross Product

the direction is given by the right-hand rule. If the vectors are expressed in terms of unit vectors i, j, and k in the x, y, and z directions, then the vector product can be expressed in the rather cumbersome form:

$$\vec{A}\times\vec{B} = \vec{i}(A_y B_z - A_z B_y) - \vec{j}(A_x B_z - A_z B_x) + \vec{k}(A_x B_y - A_y B_x)$$

The Cross Product (The Matrix)

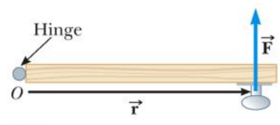


$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Force vs. Torque

- Forces cause accelerations
- Torques cause angular accelerations
- Force and torque are related

Torque



- The door is free to rotate about an axis through O
- There are three factors that determine the effectiveness of the force in opening the door:
  - The *magnitude* of the force
  - The *position* of the application of the force
  - The *angle* at which the force is applied

Torque

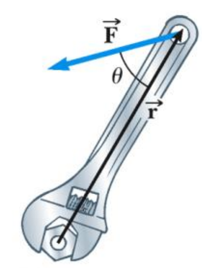
- Torque, τ, is the tendency of a force to rotate an object about some axis
  - $\tau = F \times d$
    - τ is the torque
      - symbol is the Greek letter tau
    - F is the force
    - d is the *lever arm* (or moment arm)
- Again: Torque causes objects to rotate like force causes object to move.
- SI Units
  - Newton x meter = Nm

Direction of Torque

- Torque is a vector quantity
  - The direction is perpendicular to the plane determined by the lever arm and the force
  - For two dimensional problems, into or out of the plane of the paper will be sufficient
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative

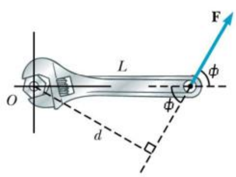
Right Hand Rule

- Point the fingers in the direction of the position vector
- Curl the fingers toward the force vector
- The thumb points in the direction of the torque



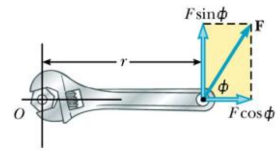
Lever Arm

- The lever arm, d, is the *perpendicular* distance from the axis of rotation to a line drawn from the axis of rotation to a line drawn along the the direction of the force
  - d = L sin Φ
  - What we are doing here is find the component of F that is perpendicular to L



An Alternative Look at Torque

- The force could also be resolved into its x- and y- components
  - The x-component, F cos Φ, produces 0 torque
  - The y-component, F sin Φ, produces a non-zero torque

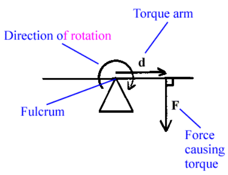


Torque, final equation

- From the components of the force or from the lever arm,

$$\tau = FL\sin\phi$$

  - F is the force
  - L is the distance along the object
  - Φ is the angle between the force and the object



Net Torque

- The net torque is the sum of all the torques produced by all the forces
  - Remember to account for the direction of the tendency for rotation
    - Counterclockwise torques are positive
    - Clockwise torques are negative

Torque and Equilibrium

- First Condition of Equilibrium
  - The net external force must be zero

$$\Sigma F = 0$$

$$\Sigma F_x = 0 \quad and \quad \Sigma F_y = 0$$

  - This is a necessary, but not sufficient, condition to ensure that an object is in complete mechanical equilibrium
  - This is a statement of translational equilibrium

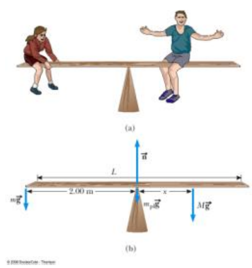
Torque and Equilibrium, cont

- To ensure mechanical equilibrium, you need to ensure rotational equilibrium as well as translational
- The Second Condition of Equilibrium states
  - The net external torque must be zero

$$\Sigma \tau = 0$$

Equilibrium Example

- The woman, mass m, sits on the left end of the see-saw
- The man, mass M, sits where the see-saw will be balanced
- Apply the Second Condition of Equilibrium and solve for the unknown distance, x



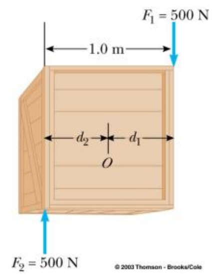
Mechanical Equilibrium

- In this case, the First Condition of Equilibrium is satisfied

$\Sigma F = 0 = 500N - 500N$

- The Second Condition is not satisfied
  - Both forces would produce clockwise rotations

$\Sigma\tau = -500Nm \neq 0$



Axis of Rotation

- If the object is in equilibrium, it does not matter where you put the axis of rotation (Pivot Point) for calculating the net torque
  - The location of the axis of rotation is completely arbitrary
  - Often the nature of the problem will suggest a convenient location for the axis
  - When solving a problem, you *must* specify an axis of rotation
    - Once you have chosen an axis, you must maintain that choice consistently throughout the problem

Center of Gravity (Center of Mass)

- The force of gravity acting on an object must be considered
- In finding the torque produced by the force of gravity, all of the weight of the object can be considered to be concentrated at one point
- In an object where all the mass is uniformly distributed, the center of mass is the geometric centroid.

Center of Gravity

When suspended from a string, an object will always rotate so that its center of gravity is directly below the string.

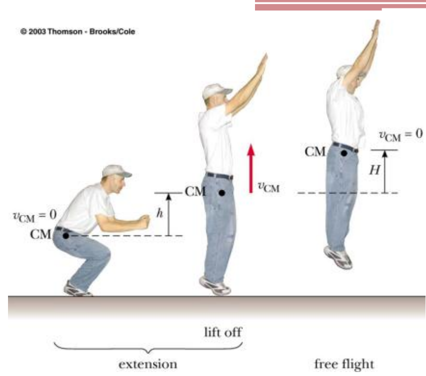
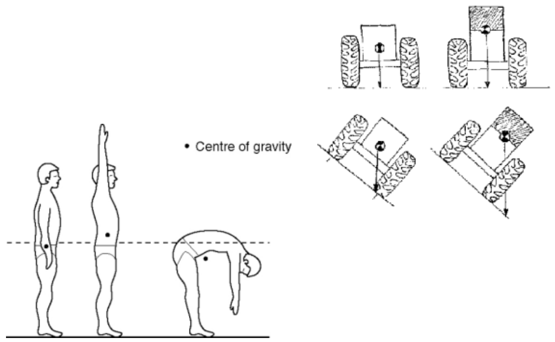


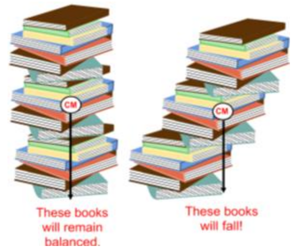
Fig. 5.27, p. 141

Slide 30



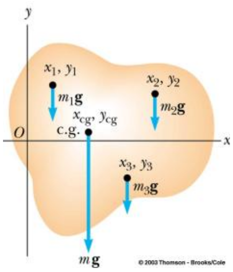
Center of Mass:

An object will balance when its center of mass is directly above its base of support.



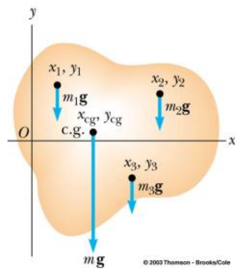
Calculating the Center of Gravity

- The object is divided up into a large number of very small particles of weight (mg)
- Each particle will have a set of coordinates indicating its location (x,y)



Calculating the Center of Gravity, cont.

- The torque produced by each particle about the axis of rotation is equal to its weight times its lever arm



Calculating the Center of Gravity, cont.

- We wish to locate the point of application of the *single force* , whose magnitude is equal to the weight of the object, and whose effect on the rotation is the same as all the individual particles.
- This point is called the *center of gravity* of the object

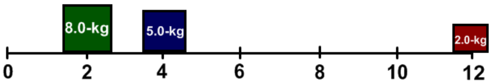
Coordinates of the Center of Gravity

- The coordinates of the center of gravity can be found from the sum of the torques acting on the individual particles being set equal to the torque produced by the weight of the object

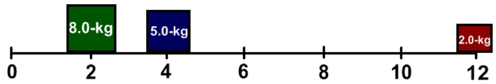
$$x_{cg} = \frac{\Sigma m_i x_i}{\Sigma m_i} \quad \text{and} \quad y_{cg} = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

Example:

An 8.0-kg mass is position at x = 2.0-m, a 5.0-kg mass is positioned at x = 4.0-m, and a 2.0-kg mass is placed at x = 12 m. Where is the center of mass of this system?



Solution:



An 8.0-kg mass is position at x = 2.0-m, a 5.0-kg mass is positioned at x = 4.0-m, and a 2.0-kg mass is placed at x = 12 m. Where is the center of mass of this system?

$$X_{CM} = \frac{(8.0)(2.0) + (5.0)(4.0) + (2.0)(12)}{8.0 + 5.0 + 2.0}$$

$$X_{CM} = 4.0 \text{ m}$$

Center of Gravity of a Uniform Object

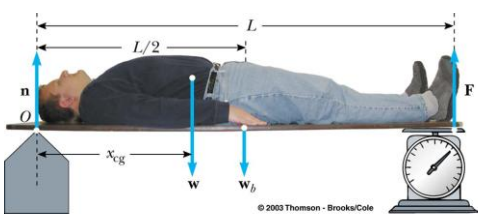
- The center of gravity of a homogenous, symmetric body must lie on the axis of symmetry.
- Often, the center of gravity of such an object is the *geometric* center of the object.

Experimentally Determining the Center of Gravity

- The wrench is hung freely from two different pivots
- The intersection of the lines indicates the center of gravity
- A rigid object can be balanced by a single force equal in magnitude to its weight as long as the force is acting upward through the object's center of gravity

Which sees more weight?

- **The person has a mass of 80Kg and the board has a mass of 5kg. The persons center of mass is 8/10 of L/2. The person is 1.7m, the board is 2m. Both the board and the person are centered on the scale and the block.**

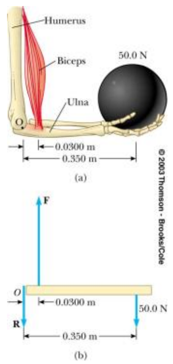


Notes About Equilibrium

- A zero net torque does not mean the absence of rotational motion
  - An object that rotates at uniform angular velocity can be under the influence of a zero net torque
    - This is analogous to the translational situation where a zero net force does not mean the object is not in motion

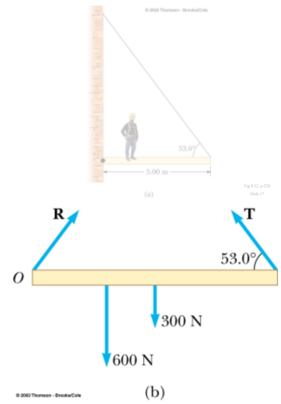
Example of a Free Body Diagram

- Isolate the object to be analyzed
- Draw the free body diagram for that object
  - Include all the external forces acting on the object

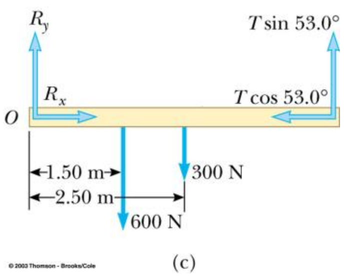


Example of a Free Body Diagram

- The free body diagram includes the directions of the forces
- The weights act through the centers of gravity of their objects

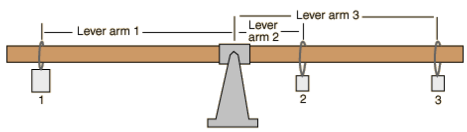


Notice the components are calculated



Try this Torque Problem (Case I)

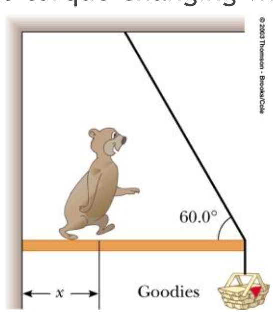
m1= 5 kg @ 2.0m
m2= 10kg @ 0.5m
m3= ??? @ 2.0m



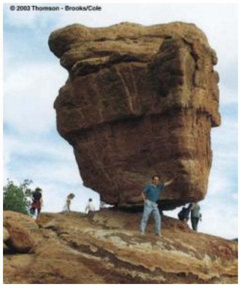
Try this Torque Problem (Case II)

The child has a mass of 60kg and the uniform board has length of 3m and is 5kg. The fulcrum is located 0.5m from the end of the board. How far away from the fulcrum must the child sit?

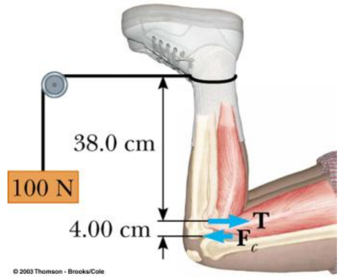
How is this torque changing with X?



Why does this bolder stay standing?



Example of Torque in Health Fields



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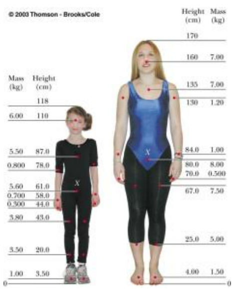
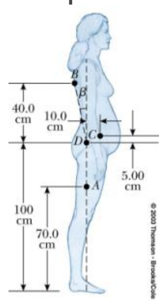
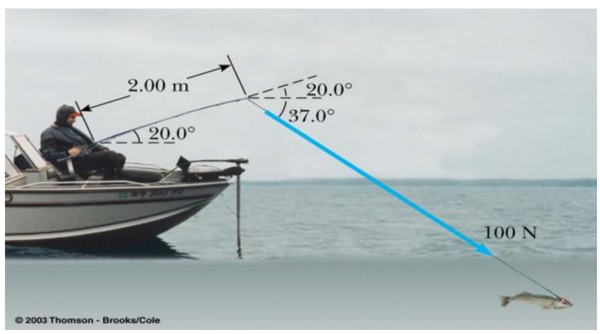


Example of Torque in Health Fields

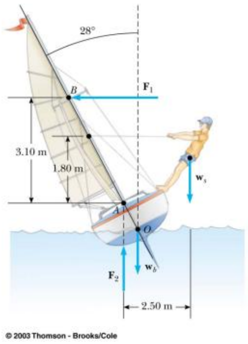


Torque in construction

Recreational Torque

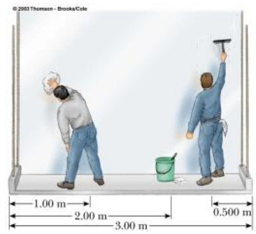


Another Recreational Torque



Try this Problem (Case III)

- **The guy on the left is 70kg and the guy on the right is 78kg. The bucket is 5kg and the scaffolding is 65kg. What is the force in each of the strings?**



Torque and Angular Acceleration

- When a rigid object is subject to a net torque (≠0), it undergoes an angular acceleration
- The angular acceleration is directly proportional to the net torque
  - The relationship is analogous to $\sum F = ma$
    - Newton's Second Law

Moment of Inertia

- The angular acceleration is inversely proportional to the analogy of the mass in a rotating system
- This mass analog is called the *moment of inertia,* I, of the object

$$I \equiv \sum m r^2$$

  - SI units are kg m²

Newton's Second Law for a Rotating Object

$$\sum \tau = I\alpha$$

- The angular acceleration is directly proportional to the net torque
- The angular acceleration is inversely proportional to the moment of inertia of the object

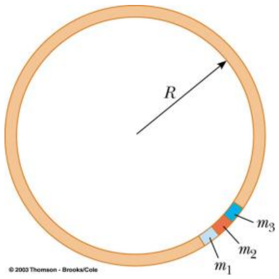
More About Moment of Inertia

- There is a major difference between moment of inertia and mass: the moment of inertia depends on the quantity of matter *and its* ***distribution*** in the rigid object.
- The moment of inertia also depends upon the location of the axis of rotation

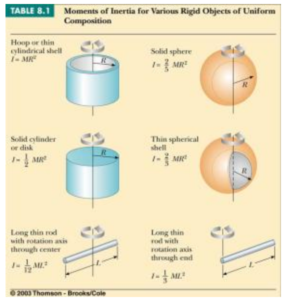
Moment of Inertia of a Uniform Ring

- Image the hoop is divided into a number of small segments, m_1 …
- These segments are equidistant from the axis

$$I = \Sigma m_i r_i^2 = M R^2$$

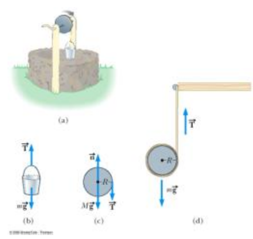


Other Moments of Inertia



Example, Newton's Second Law for Rotation

- Draw free body diagrams of each object
- Only the cylinder is rotating, so apply $\Sigma\tau = I\ \alpha$
- The bucket is falling, but not rotating, so apply $\Sigma F = m\ a$
- Remember that $a = \alpha\ r$ and solve the resulting equations



Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, ω, has rotational kinetic energy $\frac{1}{2}I\omega^2$
- Energy concepts can be useful for simplifying the analysis of rotational motion

Total Energy of a System

- Conservation of Mechanical Energy

$$(KE_t + KE_r + PE_g + PE_s)_i = (KE_t + KE_r + PE_g + PE_s)_f$$

  - Remember, this is for conservative forces, no dissipative forces such as friction can be present

Work-Energy in a Rotating System

- In the case where there are dissipative forces such as friction, use the generalized Work-Energy Theorem instead of Conservation of Energy
- $W_{nc} = \Delta KE_t + \Delta KE_R + \Delta PE$

Angular Momentum

- Similarly to the relationship between force and momentum in a linear system, we can show the relationship between torque and angular momentum
- Angular momentum is defined as
  - $L = I\omega$
  - and $\tau = \frac{\Delta L}{\Delta t}$

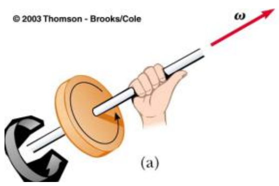
Angular Momentum, cont

- If the net torque is zero, the angular momentum remains constant
- *Conservation of Linear Momentum* states: The angular momentum of a system is conserved when the net external torque acting on the systems is zero.
  - That is, when

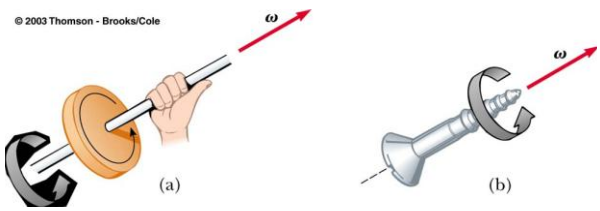
$$\Sigma\tau = 0, L_i = L_f \text{ or } I_i\omega_i = I_f\omega_f$$

Vector Nature of Angular Quantities

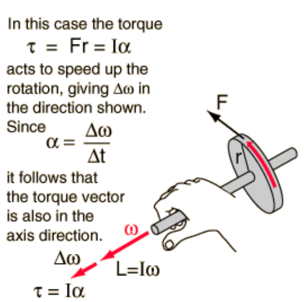
- Assign a positive or negative direction in the problem
- A more complete way is by using the right hand rule
  - Grasp the axis of rotation with your right hand
  - Wrap your fingers in the direction of rotation
  - Your thumb points in the direction of ω



Right Hand Rule / Screw Rule



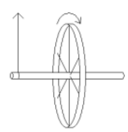
How all rotational quantities are related



From: http://hyperphysics.phy-astr.gsu.edu/hbase/rotv.html#rvec2

Precession

- Start by thinking about the object not spinning. Where would it fall.

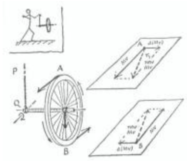


- A wheel held up by the string on the right would fall down. Produced by a torque into the screen (paper)

From: http://hyperphysics.phy-astr.gsu.edu/hbase/rotv2.html#rvec4
http://hyperphysics.phy-astr.gsu.edu/hbase/top.html#top

Precession

- Next think about the object rotating

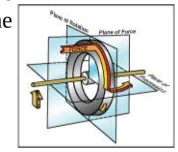


- There would be a torque pointed to the right, out of the wheel

From: http://hyperphysics.phy-astr.gsu.edu/hbase/rotv2.html#rvec4
http://hyperphysics.phy-astr.gsu.edu/hbase/top.html#top

Precession

- The difference in these two torques (one down and one to the right) is like two forces one at 90° and another at 0°. The resultant would be something in-between. The sum of these torques is something in-between. Since there is a net torque, the object has an angular acceleration in the direction of the torque.



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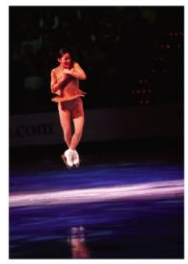
http://hyperphysics.phy-astr.gsu.edu/hbase/top.html#top

Conservation Rules, Summary

- In an isolated system, the following quantities are conserved:
  - Mechanical energy
  - Linear momentum
  - Angular momentum

Conservation of Angular Momentum, Example

- With hands and feet drawn closer to the body, the skater's angular speed increases
  - L is conserved, I decreases, ω increases



Machines

- A machine is a device that helps make work easier to perform by accomplishing one or more of the following functions:
  - transferring a force from one place to another,
  - changing the direction of a force,
  - increasing the magnitude of a force, or
  - increasing the distance or speed of a force.

Machines

Machines do not reduce the amount of work for us, but they can make it easier.

There are six types of simple machines which form the basis for all mechanical machines today.

(Simple) Machines

- The 6 simple machines are:
  - Lever
  - Inclined plane
  - Wheel and Axle
  - Pulley
  - Wedge
  - Screw

Quick Overview of Each

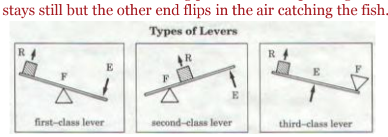
SIMPLE MACHINES	WHAT IT IS	HOW IT HELPS US WORK	EXAMPLES
LEVER	A stiff bar that rests on a support called a fulcrum	Lifts or moves loads	Shovel, nutcracker, seesaw, crow-bar, elbow
INCLINED PLANE	A slanting surface connecting a lower level to a higher level	Things move up or down it	Slide, stairs, ramp, escalator
WHEEL AND AXLE	A wheel with a rod, called an axel, through its center: both parts move together	Lifts or moves loads	Doorknob, pencil sharpener, bike

Quick Overview of Each

SIMPLE MACHINES	WHAT IT IS	HOW IT HELPS US WORK	EXAMPLES
PULLEY	A grooved wheel with a rope or cable around it	Moves things up, down, or across	Curtain rod, tow truck, mini-blind, flag pole, crane
WEDGE	A type of inclined plane with a sharp edge. The wedge moves, the inclined plane stays still.	Pushes things apart	Axe blade
SCREW	An inclined plane wrapped around a cylinder. Works with a lever.	Raises weights, presses or fastens objects	Screws, nuts

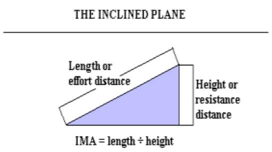
Lever

- **LEVER: The lever is a simple machine made with a bar free to move about a fixed point called a fulcrum.**
- There are three types of levers.
  - A first class lever is like a teeter-totter or see-saw. One end will lift an object (child) up just as far as the other end is pushed down.
  - A second class lever is like a wheel barrow. The long handles of a wheel barrow are really the long arms of a lever.
  - A third class lever is like a fishing pole. When the pole is given a tug, one end stays still but the other end flips in the air catching the fish.



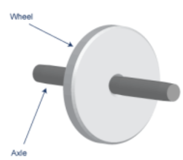
Inclined Plane

- **INCLINED PLANE : An inclined plane is a simple machine with no moving parts.**
- **It is simply a straight slanted surface.**
  - **For example: a ramp**



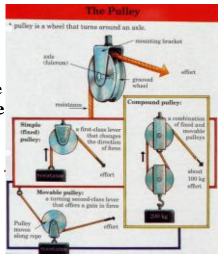
Wheel & Axle

- **WHEEL AND AXLE : A wheel and axle is a modification of a pulley. A wheel is fixed to a shaft.**
- **The wheel and shaft must move together to be a simple machine.**
  - **Sometimes the wheel has a crank or handle on it.**
  - **Examples of wheel and axles include roller skates and doorknobs.**



Pulley

- **PULLEY: A pulley is a simple machine made with a rope, belt or chain wrapped around a grooved wheel.**
- **A pulley works two ways. It can change the direction of a force or it can change the amount of force.**
  - **A fixed pulley changes the direction of the applied force. (Ex. Raising the flag) .**
  - **A movable pulley is attached to the object you are moving.**

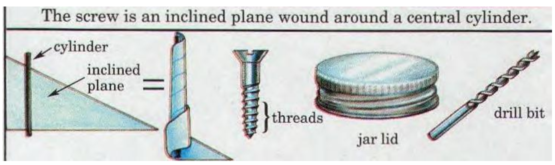


Wedge

- **WEDGE: A wedge is a modification of an inclined plane that moves .**
- **It is made of two inclined planes put together.**
  - **Instead of the resistance being moved up an inclined plane, the inclined plane moves the resistance.**

Screw

- **SCREW : A screw is a simple machine that is like an inclined plane.**
- **It is an inclined plane that wraps around a shaft.**



(Simple) Machines

- A machine can never output more work (energy) than is put into it.
- At best, $Work_{out} = Work_{in}$

Mechanical Advantage

- Machines can't multiply work or energy, but they can multiply force. ***Mechanical advantage*** measures how much a machine multiplies force.

$$MA = \frac{\text{Force machine exerts}}{\text{Force you exert}}$$

Efficiency

- The efficiency of a machine tells how much of the energy (work) that goes into the machine actually does useful work.
- It is usually expressed as a percent.

$$\textbf{Efficiency} = \frac{\textbf{Energy output}}{\textbf{Energy input}} \textbf{ x 100\%}$$